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Approximative Compactness in Orlicz Spaces

Henryk Hudzik*

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Matejki 48/49, 60-769 Poznan, Poland

and

Baoxiang Wang

Department of Mathematics, Hebei University, Baoding, 071002, People's Republic of China

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Some criteria for approximative compactness of Orlicz function and sequence spaces for both (the Luxemburg and the Orlicz) norms are presented. © 1998 Academic Press

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INTRODUCTION

Let X be a Banach space. A convex set C in X is said to be *approximatively compact* if for any $y \in X$ and any sequence $\{x_n\}$ in C which is minimizing for y, i.e., $\|y - x_n\| \to d(y, C) := \inf\{\|y - x\| : x \in C\}$, it follows that $\{x_n\}$ has a Cauchy subsequence. X is said to be *approximatively compact* if any closed convex set in X is approximatively compact (see [1]).

We discuss in this paper the approximative compactness in Orlicz function spaces L^M and Orlicz sequence spaces l^M equipped with either the Luxemburg norm or the Orlicz norm. We prove that l^M is approximatively compact if and only if it is reflexive and that L^M is approximatively compact if and only if it is reflexive and rotund (independently of the norm).

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Let $\mathbb{R}$ and $\mathbb{N}$ stand for the sets of reals and of natural numbers, respectively and let M and M^* be a couple of complementary convex and even N-functions on $\mathbb{R}$ (see [8] for the definition). Let L^0 denote the space of equivalence classes of all real measurable functions corresponding to the Lebesgue measure space (Ω, Σ, m), where $\Omega \subset \mathbb{R}$ and $m(\Omega) < \infty$. Denote by c_0 the space of all sequences with limit equal to zero. We define

$$p(x) = \rho_M(x) = \int_\Omega M(x(t))\,dt \qquad \text{and} \qquad \rho(x) = \rho_M(x) = \sum_{i=1}^{\infty} M(x(i))$$

on L^0 and c_0, respectively. Orlicz function space L^M and Orlicz sequence space l^M are defined by

$$L^M = \{x \in L^0 : \rho_M(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\},$$

$$l^M = \{x \in c_0 : \rho_M(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\}.$$

It is well known (see [2, 8–12, 15]) that L^M and l^M are Banach spaces if they are equipped with the Luxemburg norm

$$\|x\| = \|x\|_M = \inf\{c > 0 : \rho_M(x/c) \leqslant 1\}$$

or the Amemiya norm (which is equal to the Orlicz norm; see [2, 8, 12])

$$\|x\|^0 = \|x\|^0_M = \inf_{k>0} \frac{1}{k}\{1 + \rho_M(kx)\}.$$

We know (see [12, 15]) that if we define $k_x^* = \inf\{k > 0 : \rho_{M^*}(p(kx)) \geqslant 1\}$ and $k_x^{**} = \sup\{k > 0 : \rho_{M^*}(p(kx)) \leqslant 1\}$, where p denotes the right derivative of M, then

$$\|x\|^0 = \frac{1}{k}\{1 + \rho_M(kx)\}$$

for any $k \in [k_x^*, k_x^{**}]$.

We say that M satisfies the Δ_2-condition at ∞ (resp. at 0), in symbols $M \in \Delta_2^\infty$ (resp. $M \in \Delta_2^0$) if lim sup $M(2u)/M(u) < \infty$ as $u \to \infty$ (resp. $u \to 0$). M is said to be strictly convex if for all $u, v \in \mathbb{R}$ with $u \neq v$ it holds that $M((u+v)/2) < \{M(u) + M(v)\}/2$.

It is known that uniformly rotund Banach spaces are approximatively compact (see [1]). Recall that a Banach space X is said to be uniformly rotund if for all sequences $\{x_n\}$ and $\{y_n\}$ in the unit ball $B(X)$ of X it holds that $\|x_n - y_n\| \to 0$, whenever $\|x_n + y_n\| \to 2$.

RESULTS

We will prove first a lemma from which it follows that fully k-convex Banach spaces are approximatively compact. Recall that a Banach space X is said to be fully k-convex ($k \in \mathbb{N}, k \geqslant 2$) if any sequence $\{x_n\}$ in X such that $\|\sum_{i=1}^k x_{n_i}/k\| \to 1$ as $n_i \to \infty$ for $i=1, ..., k$ is a Cauchy sequence. The notation $\|\sum_{i=1}^k x_{n_i}/k\| \to 1$ as $n_i \to \infty$ for $i=1, ..., k$ means that for any $\varepsilon \in (0, 1)$ there is $m \in \mathbb{N}$ such that $\|\sum_{i=1}^k x_{n_i}/k\| > 1-\varepsilon$ whenever $n_1, ..., n_k \geqslant m$.

Lemma 1. *Let X be a Banach space. If there exists a natural number $k \geqslant 2$ such that any sequence $\{x_n\}$ such that $\|\sum_{i=1}^k x_{n_i}/k\| \to 1$ as $n_i \to \infty$ $(i=1, ..., k)$ has a Cauchy subsequence, then X is approximatively compact.*

Proof. Let C be a closed convex set in X and $x \in X \backslash C$. Let $\{x_n\}$ be a minimizing sequence for x, i.e., $\|x_n - x\| \to d(x, C) =: d$. Denote for convenience $u_n = x - x_n$ and $\lambda_n = \|u_n\|^{-1}$. Note that $x - C$ is a convex set, whence $\sum_{i=1}^k u_{n_i}/k \in x - C$. We have

$$\begin{aligned} 1 &\geqslant \left\| \sum_{i=1}^k \lambda_{n_i} u_{n_i}/k \right\| = \frac{1}{k} \left\| \sum_{i=1}^k u_{n_i}/d + \sum_{i=1}^k (\lambda_{n_i} - d^{-1})\, u_{n_i} \right\| \\ &\geqslant \frac{1}{kd} \left\| \sum_{i=1}^k u_{n_i} \right\| - \frac{1}{k} \left\| \sum_{i=1}^k (\lambda_{n_i} - d^{-1})\, u_{n_i} \right\| \\ &= \frac{1}{d} \left\| \sum_{i=1}^k u_{n_i}/k \right\| - \frac{1}{k} \left\| \sum_{i=1}^k (\lambda_{n_i} - d^{-1})\, u_{n_i} \right\|. \end{aligned}$$

Note that $\|\sum_{i=1}^k u_{n_i}/k\| \geqslant d$ and $n_i \to \infty$ implies $\lambda_{n_i} \to d^{-1} (i=1, ..., k)$. Therefore

$$\left\| \sum_{i=1}^k \lambda_{n_i} u_{n_i}/k \right\| \to 1 \qquad \text{as} \quad n_i \to \infty \ (i=1, ..., k).$$

So, the assumptions yield that $\{\lambda_n u_n\}$ has a Cauchy subsequence. It follows from the inequality

$$\|u_n - u_m\| \geqslant \frac{1}{\lambda_n} \|\lambda_n u_n - \lambda_m u_m\| + \left| \frac{\lambda_m}{\lambda_n} - 1 \right| \|u_m\|$$

that $\{u_n\}$ has a Cauchy subsequence, too. This completes the proof.

Corollary 1. *Every Banach space X which is fully k-convex for some natural $k \geqslant 2$ is approximatively compact.*

Now, we will present criteria for approximative compactness of Orlicz spaces.

THEOREM 1. *The space* $(l^M, \| \|)$ *is approximatively compact if and only if* $M \in \Delta_2^0$ *and* $M^* \in \Delta_2^0$.

Proof. It is well known that l^M is reflexive if and only if $M \in \Delta_2^0$ and $M^* \in \Delta_2^0$ (see [9, 11, 12]). Moreover, approximatively compact Banach spaces are reflexive (see [1]). So, the necessity is obvious. Now, we prove the sufficiency. By Lemma 1, we need only prove that any $\{x_n\}$ in l^M with $\|x_m + x_n\| \to 2$ as $m, n \to \infty$ has a Cauchy subsequence. First we prove that for any $\varepsilon > 0$ there exists $j_\varepsilon \in \mathbb{N}$ such that $\sum_{j=j_\varepsilon}^{\infty} M(x_n(j)) < \varepsilon$ for all $n \in \mathbb{N}$. If not, there exist $\varepsilon_0 > 0$ and two sequences $\{j_k\}$ and $\{n_k\}$ of natural numbers satisfying

$$\sum_{j=j_k}^{\infty} M(x_{n_k}(j) \geqslant \varepsilon_0. \qquad (*)$$

Since $M^* \in \Delta_2^0$, there exists $\sigma > 0$ such that (see [2, 6, 15])

$$M(u/2) \leqslant 2^{-1}(1-\sigma)\, M(u) \qquad \text{for all} \quad u \in [0, M^{-1}(1)].$$

Moreover, $M \in \Delta_2^0$ implies (see [2, 7, 15]) that for any $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x, y \in l^M$ with $\rho_M(x) \leqslant 2$ and $\rho_M(y) \leqslant \eta$, we have $|\rho(x+y) - \rho(x)| < \varepsilon$. Without loss of generality, we may assume that $\rho_M(x_n) \leqslant 2$ for all $n \in \mathbb{N}$. For any fixed $m \in \mathbb{N}$, let j_0 be sufficiently large, satisfying $\sum_{j=j_0}^{\infty} M(x_m(j)/2) < \eta$. Then for $j_k \geqslant j_0$, we have

$$\begin{aligned}
\rho_M((x_{n_k} + x_m)/2) &= \sum_{j=1}^{j_k-1} M((x_{n_k}(j) + x_m(j))/2) \\
&\quad + \sum_{j=j_k}^{\infty} M((x_{n_k}(j) + x_m(j))/2) \\
&\leqslant \sum_{j=1}^{j_k-1} 2^{-1}\{M(x_{n_k}(j)) + M(x_m(j))\} \\
&\quad + \sum_{j=j_k}^{\infty} M((x_{n_k}(j) + x_m(j))/2) \\
&\leqslant 2^{-1}\rho(x_m) + 2^{-1} \sum_{j=1}^{j_k-1} M(x_{n_k}(j)) \\
&\quad + 2^{-1}(1-\sigma) \sum_{j=j_k}^{\infty} M((x_{n_k}(j)) + \varepsilon \\
&\leqslant 2^{-1}\rho_M(x_m) + 2^{-1}\rho_M(x_{n_k}) - 2^{-1}\sigma\varepsilon_0 + \varepsilon.
\end{aligned}$$

Passing to the limit as $k, m \to \infty$, we get $1 \leqslant 1 - 2^{-1}\sigma\varepsilon_0 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this is a contradiction, which proves that condition $(*)$ holds true.

Since $M \in \Delta_2^0$ and $M^* \in \Delta_2^0$, l^M is reflexive. Hence $\{x_n\}$ has a subsequence, denoted again by $\{x_n\}$, which is weakly convergent to some x with $\|x\| \leqslant 2$. This yields that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. For any $\varepsilon > 0$ there exist $j_\varepsilon, n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{j=j_\varepsilon}^{\infty} M(x_n(j)) < \varepsilon, \qquad \sum_{j=j_\varepsilon}^{\infty} M(x(j)) < \varepsilon, \qquad \sum_{j=1}^{j_\varepsilon - 1} M((x_n(j) - x(j))/2) < \varepsilon$$

for $n \geqslant n_\varepsilon$. Thus

$$\rho_M((x_n - x)/2) \leqslant \sum_{j=1}^{j_\varepsilon - 1} M((x_n(j) - x(j))/2) + 2^{-1} \sum_{j=j_\varepsilon}^{\infty} (M(x_n(j)) + M(x(j)) < 2\varepsilon$$

for $n \geqslant n_\varepsilon$, which obviously yields that $\{x_n\}$ is a Cauchy sequence. The proof is complete.

Let p_- and p denote the left and the right derivative of M, respectively.

LEMMA 2. *Let $M \in \Delta_2^\infty$, $x \in L^M$, and $\|x\| = 1$. Then f produces a support functional at x if and only if f is of the form*

$$f(t) = w(t)/(1 + \rho_{M^*}(w)),$$

where w is a Σ-measurable function such that $p_-(x(t)) \leqslant w(t) \leqslant p(x(t))$ for m-a.e. $t \in \Omega$.

Proof. See [3, Theorem 1.3; 14, Theorem 2.1].

THEOREM 2. *The space $(L^M, \| \|)$ is approximatively compact if and only if $M \in \Delta_2^\infty$, $M \in \Delta_2^\infty$, and M is strictly convex on R.*

Proof. We know that $(L^M, \| \|)$ is fully k-convex ($k \in \mathbb{N}, k \geq 2$) if and only if $M \in \Delta_2^\infty$, $M^* \in \Delta_2^\infty$, and M is strictly convex on $\mathbb{R}$ (see [2, 4]). Hence, by Corollary 1, the sufficiency is obvious. Now, we prove the necessity. Since the approximative compactness implies reflexivity, we need only prove that M is strictly convex on $\mathbb{R}$. If not, M is affine on some interval $[a, b]$ with $0 < a < b < \infty$. We can choose a measurable closed set $E \subset \Omega$ and a measurable set $F \subset \Omega \setminus E$, both of positive measure, such that

$$2^{-1}[M(a) + M(b)]\, m(E) + M(c)\, m(F) = 1$$

for some $c>0$. We can divide E into two measurable subsets E_1^1 and E_2^1 such that $m(E_1^1)=m(E_2^1)$ and $m(E_1^1\cap E_2^1)=\emptyset$, i.e., E_1^1 and E_2^1 are disjoint up to a set of measure zero. Put

$$x_1=a\chi_{E_1^1}+b\chi_{E_2^1}+c\chi_F.$$

Repeating this procedure, we obtain a devision of E, $E=\bigcup_{i=1}^{2^n}E_i^n$, where E_i^n $(i=1, ..., 2^n)$ are pairwise disjoint sets (up to a set of measure zero) and for any fixed n they have the same measure, and $E_i^n=E_{2i-1}^{n+1}\cup E_{2i}^{n+1}$ $(i=1, ..., 2^n)$. Put

$$x_n=a\chi_{\bigcup_{k=1}^{2^{n-1}}E_{2k-1}^n}+b\chi_{\bigcup_{k=1}^{2^{n-1}}E_{2k}^n}+c\chi_F.$$

Let $C=\overline{\text{conv}}\{x_n\}$. We know by Lemma 2 that there is a common regular support functional f for all x_n $(n=1, 2, ...)$, i.e., a function f of the form from Lemma 2 such that $f\in L^{M^*}$, $\|f\|_{M^*}^0=1$, and $\langle x_n, f\rangle=\int_\Omega f(t)\,x_n(t)\,dt=1$ for $n=1, 2, ...$ Let $x\in\text{conv}\{x_n\}$, $l\in\mathbb{N}$, $a_j>0$ for $j=1, ..., l$, $\sum_{j=1}^l a_j=1$, and $x=\sum_{j=1}^l a_jx_{n_j}$. Then $\langle x, f\rangle=\sum_{j=1}^l a_j\langle x_{n_j}, f\rangle=1$. This implies that $\|x\|=1$. Thus $\|x\|=1$ for all $x\in C$. Note that

$$d(0, C)=\|x_n\|=1\qquad (n=1, 2, ...)$$

and $\rho_M(x_m-x_n)=2^{-1}M(b-a)\,m(E)$. By $M\in\Delta_2^\infty$ this yields that there exists $\sigma>0$ such that $\|x_m-x_n\|\geq\sigma$ for all $m, n\in\mathbb{N}$, which means that C is not approximatively compact. This finishes the proof.

To give a characterization of approximative compactness for L^M equipped with the Orlicz norm, we need the following lemma.

LEMMA 3. *Let $M\in\Delta_2^\infty$ and $x\in S(L^M, \|\ \|^0)$. Then $y\in L^{M^*}$ is a support functional at x if and only if:*

(i) $\rho_{M^*}(y)=1$,

(ii) *$p_-(kx(t))\leq y(t)\leq p(kx(t))$ for m-a.e. $t\in\Omega$, where k is an arbitrary fixed number from the interval $[k_x^*, k_x^{**}]$.*

Proof. See [3], Theorem 1.7.

THEOREM 3. *The space $(L^M, \|\ \|^0$ is approximatively compact if and only if $M\in\Delta_2^\infty$, $M^*\in\Delta_2^\infty$, and M is strictly convex on $\mathbb{R}$.*

Proof. We know that $(L^M, \|\ \|^0)$ is fully k-convex if and only if it is reflexive and rotund, i.e., $M\in\Delta_2^\infty$, $M^*\in\Delta_2^\infty$, and M is strictly convex on $\mathbb{R}$

(see [13]). Similarly as in Theorem 2, we need only show that the approximative compactness of $(L^M, \| \|^0)$ implies that M is strictly convex on $\mathbb{R}$. If not, there exists an interval $[a, b]$ with $0<a<b<\infty$ such that $p_-(a)=p_-(b)=p(a)=p(b)$. It is easy to see that there exists a measurable set $E\subset\Omega$ of positive measure such that $M^*(p(a))\,m(E)<1$. Moreover, we can choose $c>0$ and a measurable set $F\subset\Omega\backslash E$ such that

$$M^*(p(a))\,m(E)+M^*(p(c))\,m(F)=1.$$

Denote

$$k=1+2^{-1}[M(a)+M(b)]\,m(E)+M(c)\,m(F).$$

Similarly as in the proof of Theorem 2, there exists a decomposition of E, $E=\bigcup_{i=1}^{2^n} E_i^n$, where E_i^n are pairwise disjoint and of the same measure for any fixed $n\in\mathbb{N}$ and $E_i^n=E_{2i-1}^{n+1}\cup E_{2i}^{n+1}$ $(i=1, ..., 2^n)$. Put

$$x_n=\{a\chi_{\bigcup_{k=1}^{2^{n-1}} E_{2k-1}^n}+b\chi_{\bigcup_{k=1}^{2^{n-1}} E_{2k}^n}+c\chi_F\}/k.$$

One can easily verify that

$$\rho_{M^*}(p(kx_n))=2^{-1}[M^*(p(a))+M^*(p(b))]\,m(E)+M^*(p(c))\,m(F)=1.$$

We have that $k\in[k_{x_n}^*, k_{x_n}^{**}]$ for any $n\in\mathbb{N}$, whence

$$\|x_n\|=(1+\rho_M(kx_n))/k=\frac{1}{k}\{1+2^{-1}[M(a)+M(b)]\,m(E)+M(c)\,mF\}=1$$

for any $n\in\mathbb{N}$. Let $C=\overline{\text{conv}}\{x_n\}$. By Lemma 3, we know that there exists a function $f\in L_{M^*}$ which generates a common support functional for all x_n $(n=1, 2, ...)$. Thus, $\|x\|=1$ for all $x\in C$. Note that

$$d(0, C)=\|x_n\|=1 \qquad (n=1, 2, ...)$$

and $\rho_M(x_m-x_n)=2^{-1}M((b-a)/k)\,m(E)$. Therefore, there exists $\sigma>0$ such that $\|x_m-x_n\|\geq a$ for all $n\in\mathbb{N}$, which means that C is not approximatively compact. The proof is completed.

Remark 1. Analogous results hold true for the Lebesgue measure space (Ω, Σ, m) with $\Omega\subset\mathbb{R}$ and $m(\Omega)=\infty$. The only difference is that in place of $M\in\Delta_2^\infty$ and $M^*\in\Delta_2^\infty$ we must assume that $M\in\Delta_2$ and $M^*\in\Delta_2$, where $M\in\Delta_2$ means that there exists a positive constant K such that the inequality $M(2u)\leq KM(u)$ holds for all $u\in\mathbb{R}$.

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